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Wave Propagation in Quasi-One-Dimensional Quantum Plasma in a Magnetic Field

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A cold electron gas fills the lowest Landau level for high enough magnetic fields and for low enough densities. Such a situation is expected to occur for the Malmberg-O'Neil experiment and also for pulsar crusts and atmospheres. Such plasmas behave as a quasi-one-dimensional system and exhibit some peculiarities in their wave structure. We study the dispersion and damping of the low frequencies, i.e., the whistler mode, and the extraordinary mode for zero temperature. The behavior of the whistler mode depends critically on the "filling number" $\eta_{Fc} = \varepsilon_F/\hbar\Omega$, where ε_F is the Fermi energy and Ω is the cyclotron frequency. The one-dimensional character of the system affects the pair excitation spectrum and thus the decay of modes. We find that, in contrast to the three-dimensional situation, the plasma mode and the extraordinary mode remain undamped, while the whistler mode is undamped for all but very high k values.

1. INTRODUCTION

The properties of a nonrelativistic quantum plasma in a neutralizing background in an external magnetic field have been treated in many papers since the early 1960s. Stephen (1963) calculated the longitudinal dispersion relation for propagation vector parallel and perpendicular to a magnetic field by applying Green's function techniques. Horing (1965) analyzed this problem in more detail, particularly with respect to perpendicular propagation and damping associated with the modes. Subsequently Horing (1969) considered in detail the longitudinal static limit and the effective dielectric screening parallel and perpendicular to the field. Celli and Mermin (1964) determined the long-wavelength oscillation of a quantum plasma in a uniform magnetic field. They also calculated instability in the quantum helicon dispersion relation (V. Celli and N. D. Mermin, unpublished). Quinn and Rodriquez (1962) calculated all the elements of the dielectric tensor for spinless electrons for parallel and perpendicular propagations. The static limit of these elements was used by Quinn (1963) to calculate

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the diamagnetic susceptibility of this system. Canuto and Ventura (1972) also calculated the dielectric tensor for spinless electrons for an arbitrary angle of propagation. Kelly (1964) and Ron (1964) used a quantum kinetic approach to calculate the dielectric tensor for waves propagating along and across the magnetic field.

In our problem we treat an electron plasma in a constant, homogeneous magnetic field exactly; a perturbation approach in the photon field is used in deriving the general expressions for the dielectric tensor (Canuto and Ventura, 1972). In laboratory plasmas the magnetic field is of order 10^5 G, which is very small compared to the 10^{15} G found in pulsars. At the superstrong magnetic fields probably associated with neutron stars we have an interesting situation with the Fermi energy of the Landau levels, $P^2/2m \ll \hbar\Omega$. Only the lowest $n=0$ level is then populated, and the mobility of the electrons is therefore entirely determined by the value P_z, thus giving rise to a one-dimensional electron gas. Low density, as well as an intense magnetic filed, is necessary for this situation to be realized, since at densities $N > 10^{28} B_{12}^{3/2}$ cm^{-3}, where $B_{12} = 10^{-12}$ BG, the Fermi energy of the electrons becomes large as $\hbar\Omega$. Electron densities and fields satisfying this condition may arise in the plasma that forms the atmosphere of a neutron star.

In the Malmberg-O'Neil experiment, the magnetic field is of order 10^5 G. When this is substituted into the above condition, we find for the critical density $N_{\text{crit}} \simeq 10^7$ cm^{-3}, which is greater than the experimental density, $N_{\text{exp}} \simeq 10^{15}$ cm^{-3}. The critical temperature is $T_{\text{crit}} = \hbar\Omega/k_B \simeq 4.17 \times 10^{-4}$ K, which is also much higher than the experimental temperature. All these lead to the population of the lowest Landau level only.

In Section 2 it is shown briefly how the components of the dielectric tensor are obtained; the perturbation method used in obtaining the quantum effects on plasmas with and without damping is considered in Section 3. Sections 4 and 5 comprise the main body of the work; we determine the quantum effects on the whistler mode and the low-frequency extraordinary mode for plasmas without and with damping for arbitrary direction of propagation, respectively. In the latter case, one has to distinghish between "nonresonant" and "resonant" situations, depending on whether the cutoff frequency ω_1 is different from or coincides with the electron frequency Ω. Finally, we conclude with a brief summary of the results in Section 4.

2. DIELECTRIC TENSOR

The quantum mechanical expression for the dielectric tensor is given by (Canuto and Ventura, 1972)

$$\varepsilon_{\mu\nu}(\mathbf{k}\omega) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\delta_{\mu\nu} + \frac{\omega_p^2}{\omega^2}\tau_{\mu\nu} \tag{1}$$

where

$$\tau_{\mu\nu} = -\frac{1}{N}\sum_{mnp_xps}\left[\frac{(t_{\mu\nu})mn}{\omega - \omega_{mn}(p, p - i\hbar k_z/2) + i\eta} - \frac{(t_{\mu\nu}^{(1)})mn}{\omega + \omega_{mn}(p, p - \hbar k_z/2) + i\eta}\right] f(\varepsilon_{np}) \quad (2)$$

with

$$(t_{\mu\nu})mn = \frac{1}{M\hbar}\langle n|\Pi_{\mu}^{-k_1}|m\rangle\langle m|\Pi_{\nu}^{k_1}|n\rangle$$

$$(t^{(1)})mn = \frac{1}{M\hbar}\langle n|\Pi_{\nu}^{k_1}|m\rangle\langle m|\Pi_{\mu}^{-k_1}|n\rangle \quad (3)$$

$$\omega_{mn}\left(p, p \pm \frac{\hbar k_z}{2}\right) = (m-n)\Omega + \frac{1}{M}\left(p \pm \frac{\hbar k_z}{2}\right)k_z$$

and

$$\langle m|\Pi_x^{k_1}|n\rangle = i(\tfrac{1}{2}M\hbar\Omega)^{1/2} C_{mn} I_{mn}^{(-)}$$

$$\langle m|\Pi_y^{k_1}|n\rangle = (\tfrac{1}{2}M\hbar\Omega)^{1/2} C_{mn}\left(I_{mn}^{(+)} + \frac{ak_1}{\sqrt{2}} I_{nm}\right) \quad (4)$$

$$\langle m|\Pi_z^{k_1}|n\rangle = (p + \tfrac{1}{2}\hbar k_z) C_{mn} I_{mn}$$

$$C_{mn} = i^{m-n}\exp(-ia^2 k_1 p_x/\hbar)$$

$$I_{mn}(\rho) = (m!n!)^{-1/2}\exp(-\rho/2)\rho^{(m-n)/2}/2Q_n^{m-n}(\rho) = I_{nm}(\rho)$$

$$I_{mn}^{(+)} = n^{1/2} I_{m,n-1} + (n+1)^{1/2} I_{m,n+1}$$

$$P = P_z, \qquad k = k_z, \qquad k_1 = k_y, \qquad \rho = ak_1/\sqrt{2}, \qquad a = (\hbar/M\Omega)^{1/2}$$

The $Q_m^1(\rho)$ are associated Laguerre polynomials and $I_{mn}(\rho)$ form an orthogonal set. The integral over P in equation (2) is performed under the analyticity convention,

$$\lim_{\eta\to 0}\frac{1}{\omega + \omega_{mn} \pm i\eta} = P\frac{1}{\omega + \omega_{mn}} \mp i\pi\delta(\omega + \omega_{mn}) \quad (5)$$

Thus, the principal value of the integral in the equation gives rise to the Hermitian (refractive) part of the tensor $\tau_{\mu\nu}$, while the integral over the δ-function gives the anti-Hermitian (absorption) part.

In the degenerate limit, the distribution function $f(\varepsilon)=\theta(\varepsilon_F-\varepsilon)$ (θ is a step function) and a Fermi momentum is defined for each Landau level as

$$P_F^{(n)}=2^{1/2}m\varepsilon_F\left(1-\frac{n\hbar\Omega}{\varepsilon_F}\right)^{1/2} \tag{6}$$

and the density of electrons is given by

$$N=\frac{M\Omega}{\pi h}2\sum_{n=0}^{\infty}b_n P_F^{(n)}$$
$$=\frac{M\Omega P_0}{2\pi^2\hbar^2} \tag{7}$$

where

$$P_0=2\sum_{n=0}^{\infty}b_n P_F^{(n)} \tag{8}$$

The components of the Hermitian part of the tensor $\tau_{\mu\nu}$ for both the whistler and the extraordinary modes for any direction of propagation are calculated by applying expansions to equation (2). In the case of the whistler mode for $m=0$, $\omega/k_z\to 0$, and therefore, the equivalent of the Taylor expansion is applied, while for $m\neq 0$, $k_z\to 0$ and thus the equivalent of the asymptotic expansion is used.

In the case of the nonresonant situation of the extraordinary mode, $k_z\to 0$ for all values of m, and thus the equivalent of asymptotic expansion is applied, whereas in the case of resonant situation of the extraordinary mode, $(\omega-m\Omega)/k_z\to 0$ for all values of m, and hence the equivalent of the the Taylor expansion is applied.

3. PERTURBATION METHOD

The dispersion relation for plasma modes is given by

$$\Delta=|\boldsymbol{\varepsilon}-n^2\mathbf{T}|=0 \tag{9}$$

where

$$n=kc/\omega \qquad \text{(the refractive index)} \tag{10}$$
$$\mathbf{T}=11-\mathbf{kk}/k^2 \qquad \text{(the transverse operator)}$$

and $\boldsymbol{\varepsilon}$ is the dielectric tensor defined by equation (1). When a small perturbation is applied to the dispersion relation, we find that the frequency of the modes is shifted such that

$$\omega=\omega^0+\delta\omega; \qquad \delta\omega\ll\omega^0 \tag{11}$$

where (with θ the angle of propagation)

$$\omega^0 = \begin{cases} (\Omega_k^2 c^2/\omega_p^2) \cos \theta & \text{(whistler mode)} \\ \omega_1 & \text{(low-frequency extraordinary mode)} \end{cases} \tag{12}$$

is the unperturbed frequency, with ω_1, ω_p and Ω defined as

$$\omega_1 = \frac{\Omega}{2}\left[-1+\left(+\frac{4\omega_p^2}{\Omega^2}\right)^{1/2}\right]$$

$$\omega_p = \left(\frac{4\pi n e^2}{M}\right)^{1/2} \quad \text{(the electron plasma frequency)} \tag{13}$$

$$\Omega = \frac{eB}{Mc} \quad \text{(the electron cyclotron frequency)}$$

and

$$\delta\omega = \delta_n\omega + \delta_q\omega = -\frac{\partial\Delta_1(k\omega^0, \theta)}{\partial\Delta_0'(\omega^0, \theta)} \tag{14}$$

which is obtained by applying the Taylor expansion to the dispersion relation about ω^0. Here $\delta_n\omega$ is the frequency shift due to refractive effects and $\delta_q\omega$ is that one due to quantum effects. $\Delta_0'(\omega^0, \theta)$ is of zeroth order in k and defined as

$$\Delta_0'(\omega^0, \theta) = \left.\frac{\partial\Delta_0(\omega, \theta)}{\partial\omega}\right|_{\omega=\omega^0} \tag{15}$$

and Δ_1 is of second order in k.

4. PLASMAS WITHOUT DAMPING

As mentioned in the introduction, we determine the properties of the whistler mode and the low-frequency extraordinary mode for the cases when the propagation is parallel, perpendicular, and at an oblique angle θ to the applied external magnetic field, respectively; for perpendicular propagation neither the whistler mode nor the resonant situation of the extraordinary mode exist.

4.1. Whistler Mode

In this case we consider the frequency of the mode after perturbation to be of the form

$$\omega^2 = \omega^{02} + \delta(\omega^2) \tag{16}$$

where the second term on the right-hand side is due to both refractive and quantum effects. For parallel propagation we find that $\delta(\omega^2)$ is given by

$$\delta(\omega^2)=-\frac{2\Omega^2k^6c^6}{\omega_p^4}\left\{\left[1+\frac{\omega_p^2}{2\Omega^2}(\eta_{c0}-\eta_{F0})\right]\left[\frac{\omega_p^2}{2\Omega^2}(1-\tfrac{8}{5}\eta_{Fc})\eta_{Fc}-\tfrac{1}{2}\right]\Omega^{-2}\eta_{c0}\right.$$
$$\left.+\frac{1}{4}\left[2+\frac{6\omega_p^2}{\Omega^2}(\eta_{c0}-\tfrac{4}{3}\eta_{F0})\right]^2\frac{\omega_p^2+\Omega^2}{\omega_p^4}+(1-2\eta_{Fc})\Omega^{-2}\eta_{c0}\right\} \tag{17}$$

where

$$\eta_{Fc}=\frac{(P_F^{(0)})^2}{2m\hbar\Omega}\simeq 1.4\times10^{-3}$$
$$\eta_{F0}=P_F(0)/2m\hbar\Omega\simeq 3.2\times10^{-12} \tag{18}$$
$$\eta_{c0}=\hbar\Omega/Mc^2\simeq 4.7\times10^{-10}$$

From these equations we find that η_{F0} and η_{c0} are very small compared to η_{Fc}, and hence their contributions are negligible. Thus, equation (17) becomes

$$\delta(\omega^2)=-2\frac{\Omega^2k^6c^6}{\omega_p^4}\left\{\frac{\omega_p^2+\Omega^2}{\omega_p^4}+\frac{3}{2\Omega^2}\left[\frac{\omega_p^2+\Omega^2}{\omega_p^2}(1-\tfrac{4}{3}\eta_{Fc}+\tfrac{2}{3}(1-2\eta_{Fc})\right]\eta_{c0}\right\} \tag{19}$$

The first term in equation (19) is due to refractive effects; the second and third terms are due to quantum effects. For $\eta_{Fc}\geq\frac{3}{4}$ we have a positive quantum frequency shift. We further note that $\delta(\omega^2)$ is of order k^6.

For propagation at an oblique angle θ to the magnetic field we obtain $\delta(\omega^2)$ in the form

$$\delta(\omega^2)=-\frac{\Omega^4k^6c^6}{\omega_p^8}\left\{\left[\left(1+\frac{\omega_p^2}{\omega^2}\right)(1+\cos^2\theta)-\frac{\omega_p^2}{32\Omega^2\eta_{Fc}}\left(15-\frac{1}{2\eta_{Fc}}\right)\sin^2\theta\right.\right.$$
$$-\frac{\omega_p^2}{\Omega^2}\left[2\left\{\left[1+\frac{\omega_p^2}{\Omega^2}\left(1+\frac{\tan^2\theta}{32\eta_{Fc}^2}\right)\right]\sin^2\theta\right.\right.$$
$$\left.+\left[1+\frac{\omega_p^2}{16\Omega^2\eta_{Fc}^2}-(1-4\eta_{Fc}\tan^2\theta)\right]\cos^2\theta\right\}$$
$$-2\left\{\frac{\omega_p\sin^2\theta}{64\Omega\eta_{Fc}^2}+\frac{1}{4}\frac{\omega_p}{\Omega\eta_{Fc}}\left[(1-\tfrac{4}{3}\eta_{Fc})(1+\cos^2\theta)\right.\right.$$
$$\left.\left.+\left(1-\frac{1}{8\eta_{Fc}}\right)\sin^2\theta\right]\right\}^2$$
$$-\left(\frac{1}{2\eta_{Fc}}\left\{\left[1+\frac{\omega_p^2}{\Omega^2}\left(1+\frac{\tan^2\theta}{32\eta_{Fc}^2}\right)\right](1-\tfrac{4}{3}\eta_{Fc})\cos^2\theta\right.\right.$$

$$+\left(1+\frac{\omega_p^2}{\Omega^2}\right)\left[(1-\tfrac{4}{3}\eta_{Fc})\cos^2\theta+\left(1-\frac{1}{8\eta_{Fc}}\right)\sin^2\theta\right]$$

$$-2\frac{\omega_p^2}{\Omega^2}[(1-2\eta_{Fc})\cos^2\theta+\tfrac{3}{4}\sin^2\theta]\bigg\}$$

$$-2\left\{\left[1+\frac{\omega_p^2}{16\Omega^2\eta_{Fc}}(1-4\eta_{Fc}\tan\theta)\right]\right.$$

$$\left.+\left(1+\frac{\omega_p^2}{\Omega^2}\right)\left(1-\frac{1}{8\eta_{F_c}}\right)\tan\theta\right\}\cos^2\theta$$

$$-\frac{1}{8\eta_{Fc}^2}(1-\tfrac{4}{3}\eta_{Fc})\left[(1-\tfrac{4}{3}\eta_{Fc})\cos^2\theta+\left(1-\frac{1}{8\eta_{Fc}}\right)\sin^2\theta\right]\bigg)$$

$$+\frac{\omega_p^2}{4\eta_{Fc}\Omega^2}\left[\frac{1}{64\eta_{Fc}^2}(1-\tfrac{4}{3}\eta_{Fc})\sin^2\theta-\frac{1}{4\eta_{Fc}^2}(1-\tfrac{4}{3}\eta_{Fc})\right.$$

$$\times\left[(1-\tfrac{4}{3}\eta_{Fc})\cos^2\theta+\left(1-\frac{1}{8\eta_{Fc}}\right)\sin^2\theta\right]$$

$$-\frac{1}{2}\left\{\frac{\sin^2\theta}{16\eta_{Fc}}+\frac{1}{\eta_{Fc}}\left[(1-\tfrac{4}{3}\eta_{Fc})(1+\cos^2\theta)+\left(1-\frac{1}{8\eta_{Fc}}\right)\sin^2\theta\right]\right\}$$

$$\times\left(\left[1+\frac{\omega_p^2}{\Omega^2}\left(1+\frac{\tan^2\theta}{32\eta_{Fc}^2}\right)\right]\cos^2\theta\right.$$

$$-\frac{\omega^2p}{2\Omega^2\eta_{Fc}}\left\{\left[(1-\tfrac{4}{3}\eta_{Fc})(1+\cos^2\theta)+\left(1-\frac{1}{8\eta_{Fc}}\right)\sin^2\theta\right]\right.$$

$$\left.\left.+\left(1+\frac{\omega_p^2}{\Omega^2}\right)\right\}\right)\bigg]\bigg]\eta_{c0}\bigg\}\cos^2\theta \tag{20}$$

The contributions of η_{F0} and η_{c0} are again negligible compared to those of η_{Fc}. As a result, equation (20) becomes

$$\delta(\omega^2)=-\frac{\Omega^4k^6c^6}{\omega_p^8}\left[\left(1+\frac{\omega^2}{\Omega^2}\right)(1+\cos^2\theta)\right.$$

$$\left.-\frac{\omega_p^2}{32\Omega^2\eta_{Fc}}\left(15-\frac{1}{2\eta_{Fc}}\right)\sin^2\theta\right]\cos^2\theta \tag{21}$$

This shows that the quantum frequency shift is negative for $\eta_{Fc}>\frac{1}{30}$. As the angle of propagation θ approaches 0°, the quantum shift term of equation (21) also approaches zero and the contribution of η_{c0} in equations (20) becomes noticeable; this finally leads to equations (19) at $\theta=0°$.

4.2. Extraordinary Mode

After a small perturbation has been applied to the dispersion relation we find that the resonant situation of the extraordinary mode has a frequency shift of the form

$$\omega = \omega_1 + \delta\omega = m\Omega + \delta\omega \qquad (m = 1, 2, 3, \ldots) \tag{22}$$

whereas that for the nonresonant situation of the extraordinary mode is of the form

$$\omega = \omega_1 + \delta\omega \neq m\Omega + \delta\omega \qquad (m = 1, 2, 3, \ldots)$$

From the above we know that at resonance

$$(\omega - m\Omega)/k_z \to 0$$

for all values of m. Since we determine the frequency shift of order one, only the $m = 1$ case is considered, although for a frequency shift of order m, the general condition is still valid. In our case the cross terms of resonance and nonresonance constituting C_1 are of order $\Omega^{-2}\eta_{c0}$ smaller than the terms of order $B_0\Omega^{-2}k^2c^2$, and hence can be ignored. C_1 is of second order in k and is the component of Δ_1, i.e.,

$$\Delta_1 = B_0\Omega^{-2}k^2c^2 + C_1 \tag{24}$$

and B_0 is a function of k-independent components of the dielectric tensor. It is therefore sufficient to obtain the components of the dielectric tensor only to order zero. This allows us to have the expansions

$$\begin{aligned} \mathrm{In}|M(\omega - \Omega) \pm k_z(P_{\mathrm{F}}^{(0)} \pm kh_z/2)| &= \mathrm{In}|M\delta\omega \pm (P^{(0)} \pm \hbar k_z/2)| \\ &\simeq \pm M\delta\omega / k_z P_{\mathrm{F}}(0) \end{aligned} \tag{25}$$

for the resonant term and

$$\begin{aligned} \mathrm{In}|M(\omega + \Omega) \pm k_z(P_{\mathrm{F}}^{(0)} \pm hk_z/2)| &= \mathrm{In}|M(\delta\omega + 2\Omega) \pm k_z(P_{\mathrm{F}}^{(0)} \pm \hbar k_z/2)| \\ &\simeq \pm P_F^{(0)} k_z/2M\Omega \end{aligned} \tag{26}$$

for the nonresonant term of the resonant situation of the extraordinary mode, respectively.

Nonresonant case: For propagation parallel to a magnetic field we find that the frequency shift is given by

$$\delta\omega = \frac{k^2c^2}{2\omega_1 + \Omega}\left[\frac{(\omega_1 + \Omega)^2}{\omega_p^2} + \left(1 - \frac{4}{3}\frac{\Omega}{\omega_1 + \Omega}\eta_{\mathrm{F}c}\right)\eta_{c0}\right] \tag{27}$$

This shows that resonance does not exist for parallel propagation. It can also be seen that the frequency shift due to quantum effects depends on $\eta_{\mathrm{F}c}$ and is positive for all values of $\eta_{\mathrm{F}c}$.

When the direction of propagation is perpendicular to the magnetic field the frequency shift is of the form

$$\delta\omega = \frac{k^2c^2}{2\omega_1+\Omega}\left[\frac{(\omega_1+\Omega)^2}{2\omega_p^2}+\frac{\omega_1+\Omega}{\omega_1+2\Omega}\,\eta_{c0}\right] \tag{28}$$

Equation (28) shows that resonance does not exist in order k^2. But the k^4 term, which is due to a second-order frequency shift, has a resonance at $\omega_1=\Omega$ and $\omega_1=2\Omega$ as in classical plasmas. Thus, as we approach the neighborhood of Ω and 2Ω, the second-order frequency shift becomes infinitely large and we cannot ignore its contribution.

Turning now to the situation in which the direction of propagation is at an arbitrary angle θ to the magnetic field, we find the frequency shift

$$\delta\omega = \frac{k^2c^2}{2(2\omega_1+\Omega)}\left\{\frac{(\omega_1+\Omega)^2}{\omega_p^2}(1+\cos^2\theta) + \left[\left(1-\frac{4\Omega}{3(\omega_1+\Omega)}\,\eta_{\mathrm{F}c}\right)\cos^2\theta+\frac{2(\omega_1+\Omega)}{\omega_1+2\Omega}\sin^2\theta\right]\eta_{c0}\right\} \tag{29}$$

This leads us to conclude that resonance does not show up in the k^2 term. However, as in classical plasmas, in the neighborhood of Ω and 2Ω the second-order frequency becomes infinitely large and its effects cannot be ignored. We also note that the quantum frequency shift is positive for all values of $\eta_{\mathrm{F}c}$ and θ.

Resonant case: When the direction of propagation is parallel to the magnetic field the frequency shift is given by

$$\delta\omega = (4\Omega^2-3\omega_p^2)\,\frac{\Omega^{-1}\varepsilon_{\mathrm{F}}}{\omega_p^2 M}\,k^2 \tag{30}$$

However, equation (27) shows that resonance does not exist in this situation. This implies that equation (30) is not the frequency shift of the resonant situation of the extraordinary mode, but that of the cyclotron mode.

In the case where the direction of propagation is at an oblique angle θ to the magnetic field we find that the frequency shift is given by

$$\delta\omega = \left[(4\Omega^2-3\omega_p^2)\cos^2\theta+\frac{\omega_p^4}{\omega_p^2-\Omega^2}\sin^2\theta\right]\frac{\Omega^{-1}\varepsilon_{\mathrm{F}}}{\omega_p^2 M} \tag{31}$$

It is already known from the above that the frequency shift of order lower than k^4 does not have a resonance; we therefore conclude that equation (31) is not the frequency shift of the resonant situation of the extraordinary mode, but that of the cyclotron mode. It is also known that at resonance, $\omega_p>\Omega$, and therefore the third term is positive in that situation.

5. PLASMAS WITH DAMPING

From equation (2) we obtain the anti-hermitian part of $\tau_{\mu\nu}$ as

$$\tau^A_{\mu\nu} = \frac{i\pi}{N} \sum_{mnP_xPs} b_n f(E_{np})(t_{\mu\nu})_{mn}[\delta(\omega-\omega_{mn})-\delta(\omega+\omega_{mn})]$$

$$= \frac{i\pi}{N} \sum_{mn} b_n \int_{-\infty}^{\infty} \frac{M\Omega}{2(\pi\hbar)^2}\, dp\, f(E_{np})[\delta(\omega-\omega_{mn})-\delta(\omega+\omega_{mn})]$$

$$= \frac{i\pi M}{P_0 N k_z} \sum_{mn} b_n (\tau_{\mu\nu})_{mn} \left[\theta\left(\left(\frac{\omega}{k_z} + \frac{k_z\hbar}{2M} + \frac{\Omega}{k_z^2}\right)^2 - \frac{1}{M^2}(P_F^{(n)})^2 \right) \right.$$

$$\left. - \theta\left(\left(\frac{\omega}{k_z} - \frac{k_z h}{2M} - \frac{\Omega}{k_z}\right)^2 - \frac{1}{M^2}(p_F^{(n)})^2 \right) \right] \tag{32}$$

Equation (32) shows that the damping domain is given by

$$\omega_{\min} < \omega < \omega_{\max} \tag{33}$$

where

$$\omega_{\min} = -\frac{k_z p_F^{(n)}}{M} - \frac{k_z^2\hbar}{2M} - l\Omega$$
$$\omega_{\max} = k_z \frac{P_F^{(n)}}{M} - \frac{k_z^2\hbar}{2M} - l\Omega \tag{34}$$

for the first term of the equation and

$$\omega_{\max} = -\frac{k_z P_F^{(n)}}{M} + \frac{k_z^2\hbar}{2M} + l\Omega$$
$$\omega_{\max} = -\frac{k_z P_F^{(n)}}{M} + \frac{k_z^2\hbar}{2M} + l\Omega \tag{35}$$

for the second term of the equation. In our case $n = 0$; therefore, equations (34) and (35) become

$$\omega_{\min} = -\frac{P_F^{(0)} k_z}{M} - \frac{\hbar k_z^2}{2M} - m\Omega$$
$$\omega_{\max} = -\frac{P_F^{(0)} k_z}{M} + \frac{\hbar k_z^2}{2M} - m\Omega \tag{36}$$

and

$$\omega_{\min} = -\frac{P_F^{(0)} k_z}{M} + \frac{\hbar k_z^2}{2M} + m\Omega$$
$$\omega_{\max} = \frac{P_F^{(0)} k_z}{M} + \frac{\hbar k_z}{2M} + m\Omega \tag{37}$$

respectively.

5.1. Whistler Mode

In this case, we consider only $m=0$ contribution as $k \to 0$ as in the undamped plasmas. As can be seen from Figure 1, this is the only equation that could give a contribution. This causes equations (36) and (37) to become

$$\omega_{\min} = -\frac{P_F^{(0)} k_z}{M} - \frac{k_z^2 \hbar}{2M}$$
$$\omega_{\max} = \frac{P_F^{(0)} k_z}{M} - \frac{k_z^2 \hbar}{2M} \tag{38}$$

and

$$\omega_{\min} = -\frac{P_F^{(0)} k_z}{M} + \frac{\hbar k_z^2}{2M}$$
$$\omega_{\max} = \frac{P_F^{(0)} k_z}{M} + \frac{\hbar k_z^2}{2M} \tag{39}$$

respectively.

For parallel propagation, $(t_{\mu\nu})_{mn}$ becomes $\delta_{m,n+1}$. But, as already stated, only $m=0$ contributes. This implies that there are no damping effects in this situation. For oblique propagation there could be damping. However, from Figure 1 we find that the damping effects cancel out in regions denoted by O_i ($i=1,2,3,\ldots$). But the whistler mode is the region O_1 and O_2; hence we conclude that there is no damping effects on the whistler mode. This feature is the result of the quasi-one dimensional character of the electron distribution.

5.2. Extraordinary Mode

Nonresonant case. From equations (36) and (37) we find that the nonresonant situation exists for $m \neq 1, 2, 3, \ldots$. As a result of this requirement we find that the components of the anti-Hermitian part of $\tau_{\mu\nu}$ vanish. Therefore, we conclude that there is no damping effect on waves in this case.

Resonant case. The components of dielectric tensor are obtained by considering the $m=1$ situation to the lowest order in k in any direction of propagation. By going through the same procedure as that for undamped plasmas, we obtain the dispersion relation for parallel propagation to be of the form

$$1 - \frac{3}{4}\frac{\omega_p^2}{\Omega^2} + \frac{\omega_p^2 M}{2\Omega P_0 k}\left[i\pi + \ln\left| \frac{M\delta\omega - kP_F^{(0)}}{M\delta\omega + kP_F^{(0)}} \right| \right] = 0 \tag{40}$$

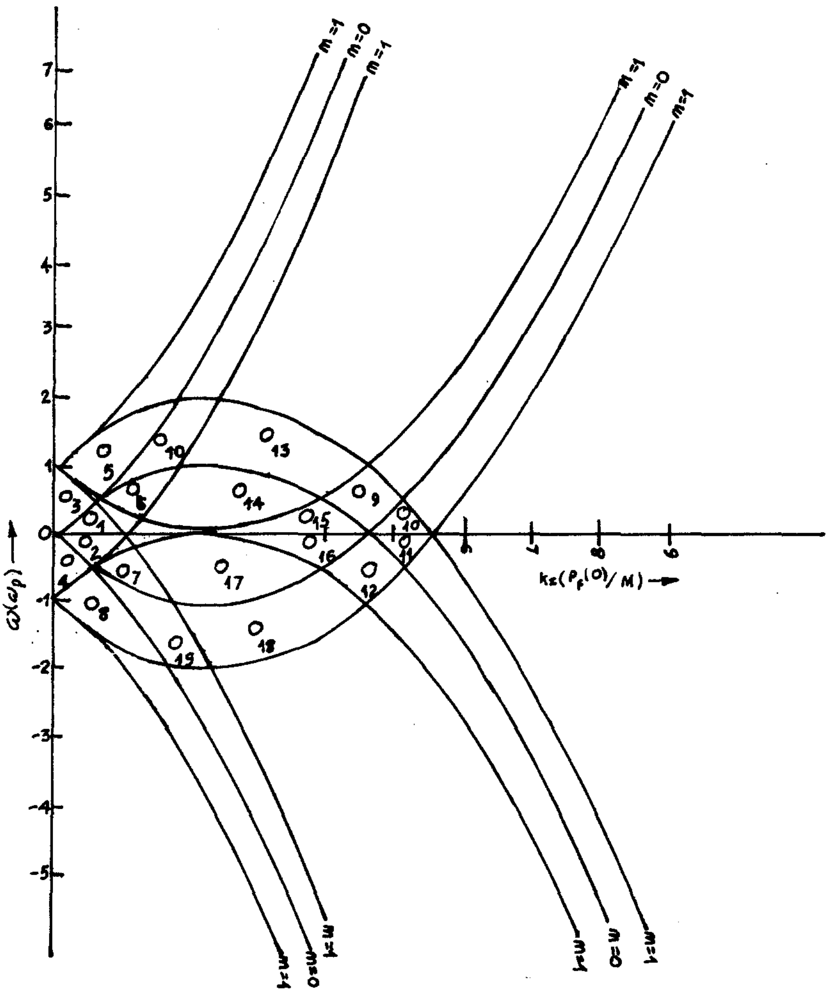

Fig. 1. Pair excitation boundaries, determining damping: ω (in units of ω_p) versus k_z (in units of $P_F^{(0)}/M$). There is no damping in regions denoted by O_i, where $i = 1, 2, 3, \ldots$. The whistler mode exists in regions O_1 and P_2.

so long as

$$\delta\omega < P_F^{(0)} k/M \tag{41}$$

which can also be expressed as

$$1 - \frac{3\omega_p^2}{4\Omega^2} + \frac{\omega_p^2}{2\Omega P_0 k} \ln \frac{M\delta\omega - P_F^{(0)} k}{M\delta\omega + P_F^{(0)} k} = 0 \tag{42}$$

The imaginary term in equation (40) is absorbed in equation (42). If equation (40) is expanded in $\delta\omega$, we obtain

$$\delta\omega \doteq \frac{2\Omega k^2}{\kappa^2}\left[\left(1-\frac{3\omega_p^2}{4\Omega^2}\right)+i\pi\omega_p\frac{\kappa}{k}\right] \tag{43}$$

where

$$\kappa^2 = M\omega_p^2/2\varepsilon_F \tag{44}$$

This is not true, since there is no instability in an equilibrium system which is being considered. In order to get the right solution of equation (40), we solve it without expanding it in $\delta\omega$, as follows:

$$\ln\frac{M\delta\omega - P_F^{(0)}k}{M\delta\omega + P_F^{(0)}k} = -\frac{4\Omega}{\omega_p}\left(1-\frac{3\omega_p^2}{4\Omega^2}\right)\frac{k}{\kappa} \tag{45}$$

Thus

$$\delta\omega = 2^{1/2}\Omega\Lambda(\lambda)k/\kappa \tag{46}$$

where

$$\Lambda(\lambda) = \frac{1+\exp(-\lambda k\kappa^{-1})}{1-\exp(-\lambda k\kappa^{-1})} \tag{47}$$

where

$$\lambda = \frac{4\Omega}{\omega_p}\left(1-\frac{3\omega_p^2}{4\Omega^2}\right) \tag{48}$$

From equation (54) we find that

$$\begin{aligned}\Lambda(\lambda) &> 1: \qquad \lambda > 0\\ &< 1: \qquad \lambda < 0\end{aligned} \tag{49}$$

Figure 2 shows that $\delta\omega$ lies outside the damping domain given by Figure 1. This allows us to conclude that the cyclotron mode is not damped in this case. Further, for $\lambda k\kappa^{-1} \ll 1$ we recover the result for plasmas without damping.

For propagation at an arbitrary angle θ to the magnetic field the dispersion relation after a small perturbation is applied in the vicinity of the resonance is given by

$$\left(1-\frac{3}{4}\frac{\omega_p^2}{\Omega^2}\right)\left(1-\frac{\omega_p^2}{\Omega^2}\right)-\frac{\omega_p^4}{4\Omega^4}\tan^2\theta+\frac{\omega_p}{4\Omega}\left(1-\frac{\omega_p^2}{\Omega^2}\right)\frac{\kappa}{k_z}\ln\frac{M\delta\omega - P_F^{(0)}k_z}{M\delta\omega + P_F^{(0)}k_z} \tag{50}$$

This leads to

$$\delta\omega = 2^{1/2}\Omega\Lambda(\bar{\lambda})\frac{k}{\kappa}\cos\theta \tag{51}$$

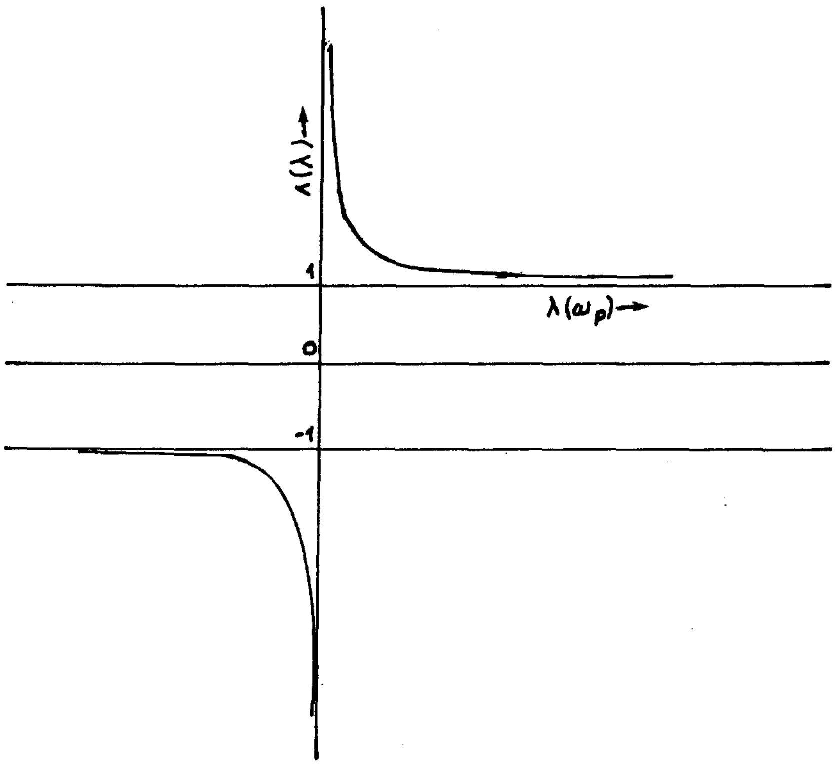

Fig. 2. The function Λ (in units of λ) versus λ (in units of ω_p). See text.

where

$$\bar{\lambda} = \lambda \cos\theta + \frac{\omega_p^4}{4\Omega^2(\omega_p^2 - \Omega^2)} \sin^2\theta \sec\theta \tag{52}$$

Equation (49) is still satisfied for $\bar{\lambda}$; this implies that Figure 2 is also valid for $\bar{\lambda}$. Since Figure 2 shows that $\delta\omega$ lies outside the damping domain given by Figure 1, we conclude that there is no damping effect on the cyclotron mode, as we are already aware that there is no resonance for the frequency shift of order lower than k^4. Also, for $\bar{\lambda} k \bar{\kappa}^1 \ll 1$ the result for plasmas without damping is recovered.

6. CONCLUSION

We find that the whistler mode is undamped in any direction of propagation for all but very high k values. This is a consequence of the one-dimensional character of the situation. The frequency shift due to quantum effects is found to be of order k^6 and angle-dependent, and is either positive or negative, depending on the value of η_{Fc}, the ratio of Fermi energy to the level separation.

We also find that the extraordinary mode is undamped in any direction of propagation. For nonresonance the frequency shift due to quantum effects is of order k^2; it is also η_{Fc}- and angle-dependent. We further find that resonance exists for a frequency shift due to quantum effects of order k^4 and higher powers of k and for oblique propagation; the frequency shift of order k^2 that is formally obtained in this case is the frequency of an independent cyclotron mode. The cyclotron mode is also seen to be η_{Fc}- and angle-dependent.

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